Point-Based Methods for Estimating Curve Length and Surface Area

Michael S. Floater Atgeirr F. Rasmussen

CMA, University of Oslo

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Mathematics for
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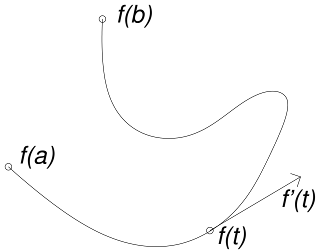
Definition of curve length

Given a parametric curve segment $\mathbf{f} : [a, b] \to \mathbb{R}^d$, its length is given by the formula

$$L(\mathbf{f}) = \int_a^b |\mathbf{f}'(t)| \, dt.$$

We assume that $\mathbf{f}'(t)$ exists, is continuous, and that $\mathbf{f}$ is regular in $[a, b]$.



Computing curve length

In general, the integral cannot be computed analytically, even for polynomial curves. (Feasible only for very low degree.)

We must use numerical integration.

Using numerical integration

We define the 'speed' function

$$F(t) := |\mathbf{f}'(t)|, \qquad a \leq t \leq b,$$

and apply some standard quadrature rule to F:

$$\int_a^b F(t)\,dt \approx \sum_{i=0}^{n} w_i F(t_i),$$

for some nodes

$$a \leq t_0 < t_1 < \cdots t_n \leq b,$$

and weights $w_0, w_1, \ldots, w_n$, giving the estimate

$$L(\mathbf{f}) \approx \overline{L(\mathbf{f})} := \sum_{i=0}^{n} w_i |\mathbf{f}'(t_i)|.$$

Drawbacks

The integral can be evaluated very efficiently using for instance Gaussian quadrature, but the method has some drawbacks:

- The derivatives $\mathbf{f}'(t_i)$ may be expensive
- The derivatives may not be available
- For high order methods, the curve must have high smoothness

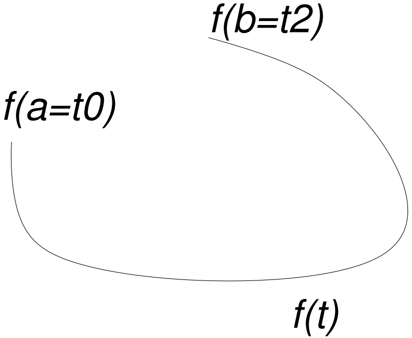
Two-stage method features

The two-stage method has the following features:

- Only point evaluation needed
- May obtain rules of arbitrarily high order
- Need one less degree of smoothness than derivative-based methods

First stage

First we interpolate $\mathbf{f}$ at an increasing sequence of nodes t_i, by a polynomial $\mathbf{p}_n : \mathbb{R} \to \mathbb{R}^d$ of degree $\leq n$, thus $\mathbf{p}_n(t_i) = \mathbf{f}(t_i)$ for $0 \leq i \leq n$.



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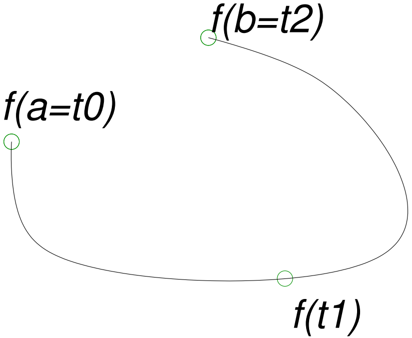

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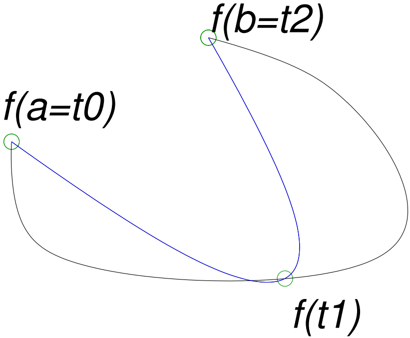
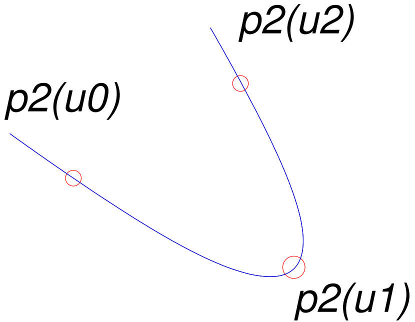


Second stage

Then we apply some quadrature rule

$$L(\mathbf{p}_n) \approx \overline{L(\mathbf{p}_n)} := \sum_{j=0}^{m} w_j P_n(u_j)$$

to the speed function $P_n(t) := |\mathbf{p}_n'(t)|$.



Two-stage method notes

Some comments on the two-stage method:

- The interpolation nodes t_i need not be related to the quadrature nodes u_j. A priori we have full freedom to choose both.
- The length of $\mathbf{p}_n$ approximates the length of $\mathbf{f}$ up to a certain order. The order will depend on the degree n of $\mathbf{p}_n$ and on the choice of interpolation points t_i.
- For particular choices of interpolation points and quadrature methods, we reproduce some earlier methods, such as chord length, or the method of Vincent and Forsey.

Approximation theorem

Theorem

Suppose $\{t_i\}, i = 0, \ldots, n$ are the $n+1$ Gauss-Lobatto points in the interval $[a, b]$. If $\mathbf{f} \in C^{2n}[a, b]$ and $\mathbf{f}$ is regular in $[a, b]$ then

$$|L(\mathbf{f}) - L(\mathbf{p}_n)| \in O(h^{2n+1})$$

as $h \to 0$.

The total error

To study the total error observe that

$$|L(\mathbf{f}) - \sum_{i=0}^{m} w_i P_n(u_i)| \leq |L(\mathbf{f}) - L(\mathbf{p}_n)| + |L(\mathbf{p}_n) - \sum_{i=0}^{m} w_i P_n(u_i)|,$$

The first term is bounded by our theorems, and the second by the error in the quadrature rule used. Assuming that we use a quadrature rule with an $O(h^{2n+1})$ error term, and that the interpolation points are chosen according to the second theorem, the total error is also $O(h^{2n+1})$ for sufficiently small h.

Note that if we use the method in a composite fashion, this is the local error. The global error will be $O(h^{2n})$.

Degree one

For $n = 1$ there is only one possible choice of the points t_0, t_1 which satisfies the conditions of either theorem. We must set

$$t_0 = a$$
$$t_1 = b$$

With any reasonable quadrature this gives the chordlength rule:

$$\overline{L(\mathbf{p}_1)} = |\mathbf{f}(b) - \mathbf{f}(a)|$$

and our analysis shows that

$$|L(\mathbf{f}) - \overline{L(\mathbf{p}_1)}| = O(h^3) \qquad \text{as } h \to 0.$$

Degree two (I)

For $n = 2$ there is precisely one choice of the points t_0, t_1, t_2 which satisfies the conditions of the second theorem. We must set

$$\begin{aligned} t_0 &= a \\ t_1 &= \frac{a+b}{2} \\ t_2 &= b \end{aligned}$$

With this choice, if $f \in C^4[a,b]$ then

$$|L(\mathbf{f}) - L(\mathbf{p}_2)| \leq Ch^5.$$

Degree two (II)

Now we consider two choices of quadrature rules in order to achieve an $O(h^5)$ rule for $L(\mathbf{f})$.

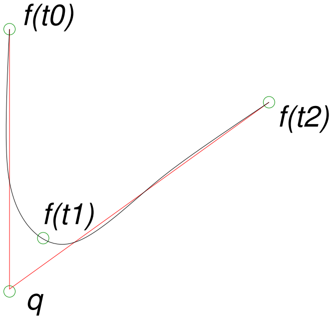
- The two-point Gauss rule gives

$$\overline{L(\mathbf{p}_2)} = \frac{(b-a)}{2}(|\mathbf{p}_2'(u_0)|+|\mathbf{p}_2'(u_1)|),$$

where u_0, u_1 are $\frac{a+b}{2} \mp \frac{b-a}{6}\sqrt{3}$. This rule can be written in the form

$$\overline{L(\mathbf{p}_2)} = |\mathbf{f}(t_0) - \mathbf{q}| + |\mathbf{q} - \mathbf{f}(t_2)|$$

where $\mathbf{q}$ is a point that depends linearly on $\mathbf{f}(t_0)$, $\mathbf{f}(t_1)$ and $\mathbf{f}(t_2)$.



Degree two (III)

- A second choice gives a very simple rule in terms of the points $\mathbf{f}(t_i)$, $i = 0, 1, 2$. We use the three-point open Newton-Cotes formula with nodes $\frac{3a+b}{4}$, $\frac{a+b}{2}$ and $\frac{a+3b}{4}$. Then

$$\overline{L(\mathbf{p}_2)} = \frac{(b-a)}{3}\left(2|\mathbf{p}_2'(u_0)| - |\mathbf{p}_2'(u_1)| + 2|\mathbf{p}_2'(u_2)|\right).$$

  Since the quadrature rule is $O(h^5)$, so is this rule. This can be rewritten as

$$\frac{4}{3}\Big(|\mathbf{f}(t_1) - \mathbf{f}(a)| + |\mathbf{f}(b) - \mathbf{f}(t_1)|\Big) - \frac{1}{3}|\mathbf{f}(b) - \mathbf{f}(a)|,$$

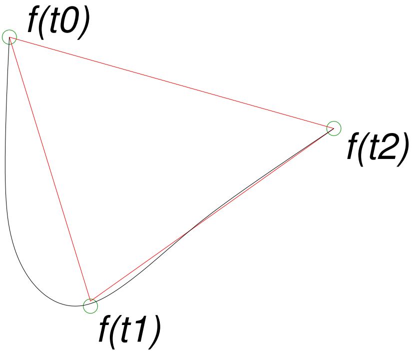
  which is simply a linear combination of the lengths of a fine polygon and a coarse polygon.

Degree two (IV)

This turns out to be the same as Vincent and Forsey's method.

However, the proof of approximation order is new.

It can also be deduced by using the idea of Richardson extrapolation, but proving it is a bit more work.



Surface area

Extending the idea to compute areas by point evaluation only, we have tried to prove theorems like before.

We have succeeded on rectangular domains. The basic approach is using a tensor-product variant of the curve case.

Computing area

Given a parametric surface $\mathbf{s} : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$, we can compute its area:

$$A(\mathbf{s}) = \int_{[a,b]\times[c,d]} |\mathbf{s}_u(u, v) \times \mathbf{s}_v(u, v)| dudv$$

The two-stage method, revisited

Given interpolation nodes $\{u_i\}$ in $[a, b]$ and $\{v_j\}$ in $[c, d]$, we let $\mathbf{p}_n$ now be the bidegree (n, n) polynomial interpolating $\mathbf{s}$ at the points (u_i, v_j) where $i, j = 0, \ldots, n$.

The question is: To what order will $A(\mathbf{p}_n)$ approximate $A(\mathbf{s})$? Now $h := \max(b - a, d - c)$.

Area approximation theorem

Theorem

Suppose $\{u_i\}$ *and* $\{v_j\}$ *are the* $n+1$ *Gauss-Lobatto points in the intervals* $[a,b]$ *and* $[c,d]$*, respectively. If* $\mathbf{s} \in C^{2n,2n}$ *and* $\mathbf{s}$ *is regular in* $[a,b] \times [c,d]$ *then*

$$|A(\mathbf{s}) - A(\mathbf{p}_n)| \in O(h^{2n+2})$$

as $h \longrightarrow 0$*.*

Example

The degree $(1, 1)$ case: The interpolation polynomial is simply the bilinear polynomial interpolating at the four corners of the quad.

Using the tensor-product midpoint rule is sufficiently precise.

One can show that this gives the familiar formula

$$\overline{A(\mathbf{s})} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

where $\mathbf{a}, \mathbf{b}$ are the diagonals of the quad. It is thus shown to be a second order method (globally, as a composite method).

Richardson extrapolation

Richardson extrapolation works very well for curve lengths, and gives two orders every time! A proof of this is forthcoming.

Richardson extrapolation seems to work for areas, but only gives one order improvement. No proof yet.

A nice complementary technique. Less efficient for high orders, but easy to implement.

Are all RE methods in the two-stage family? Open question.

Other remarks

- For areas, triangle domains are harder than tensor-product domains and polynomials. No strong results yet.
- Technique generalizes to all integrals of the form $\int_a^b G(g, g')dt$ (use Gauss-Lobatto points).
- For higher derivative generalizations such as $\int_a^b G(g, g', g'')dt$, you need some derivatives for optimal method order. That is, the polynomial is found by Hermite interpolation.
- Article dealing with the curve case to be printed in JCAM.